

The Proof of the Collatz Conjecture

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Abstract

In this paper, I intend to prove the Collatz conjecture. In the following, I will prove that the Collatz conjecture is true for any positive integer and also for natural numbers. That is, after sufficiently using the rules of the Collatz conjecture, we must reach 1. In fact, I aim to give a definite answer to the question that has not been answered since 1937: Does the Collatz sequence eventually reach 1 for all-natural or positive integer initial values? My answer is yes.

Keyword: number theory, Collatz conjecture, Collatz sequence, number Set, theory

Introduction

The Collatz conjecture proposed by Lothar Collatz, a German mathematician at the Humboldt University of Berlin, in 1937, is one of the unsolved problems in number theory in mathematics. According to the Collatz conjecture, n is an arbitrary positive integer or a natural number ($n \neq 0$, $n \in \mathbb{Z}$ or $n \in \mathbb{N}$). If n is an even number, divide it by 2; otherwise, multiply it by 3, add it to 1, and divide it by 2, and then, repeat the operations and processes, i.e. taking the result of each step as the input for the next step. In this way, one can create a sequence of numbers, beginning with a positive integer regardless of the n value, which will always reach 1.

In fact, the Collatz conjecture is a recursive and a piecewise function, $f(n) = \text{FSc}(n)$, mapping from $\mathbb{N} \vee \mathbb{Z}^+$ onto $\mathbb{N} \vee \mathbb{Z}^+$; that is,

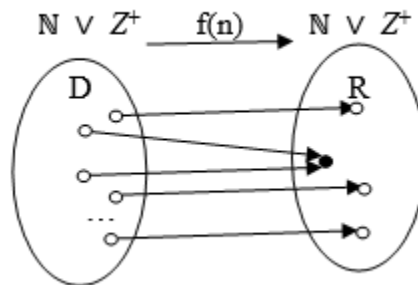


Figure 1. Mapping from $\mathbb{N} / \mathbb{Z}^+$ onto $\mathbb{N} / \mathbb{Z}^+$ by the Collatz conjecture relation

The Collatz conjecture in mathematics formula is as follows:

$$f(n) = \begin{cases} n / 2 & ; \text{if } n \text{ is even} \\ (3n + 1) / 2 & ; \text{if } n \text{ is odd} \end{cases}$$

($n / 2$ and $(3n + 1) / 2$ are even numbers)

where $f(n)$ is the relation in the rules of the Collatz conjecture.

Thus, the above formula can be rewritten as follows:

1. Transferring to a greater number/increasing the value/amplifying the

amplitude/having a delay in reaching 1 $\Rightarrow 3n$

2. The odd to even conversion $\Rightarrow (3n + 1) \Rightarrow$

3. Division of the result by 2 $\Rightarrow (3n + 1) / 2 \Rightarrow$ (repetition of the procedure for odd numbers).

Some examples of the Collatz conjecture as a piecewise function mapping, $F_{Sc}(n)$, are as follows:

$$F_{Sc}(1) = \{(1,2), (2,1)\}$$

$$F_{Sc}(2) = \{(2,1)\} *$$

$$F_{\text{Sc}(3)} = \{(3,5), (8,4), (4,2), (2,1)\}$$

$$F_{\text{Sc}(4)} = \{(4,2), (2,1)\}$$

$$F_{\text{Sc}(5)} = \{(5,8), (8,4), (4,2), (2,1)\}$$

$$F_{\text{Sc}(6)} = \{(6,3), (5,8), (8,4), (4,2), (2,1)\}$$

$$F_{\text{Sc}(7)} = \{(7,11), (17,26), (26,13),$$

$$(40,20), (20,10), (5,8), (8,4), (4,2), (2,1)\}$$

$$F_{\text{Sc}(8)} = \{(8,4), (4,2), (2,1)\}$$

$$F_{\text{Sc}(9)} = \{(9,14), (14,7), (7,11), (17,26), (26,13),$$

$$(40,20), (20,10), (5,8), (8,4), (4,2), (2,1)\}$$

$$F_{\text{Sc}(10)} = \{(10,5), (5,8), (8,4), (4,2), (2,1)\}$$

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The following definitions and symbols are necessary, as I will use them

throughout the paper.

\mathbb{Z} = The set of integer numbers

\mathbb{Z}^+ = The set of positive integer numbers

\mathbb{N} = The set of natural numbers

$\text{Scc}(n)$ = The Collatz conjecture sequence for $n \in \mathbb{N}$ or $n \in \mathbb{Z}^+$, \mathbb{Z} , $n \neq 0$

$\mathbb{N} = \{1,2,3,4,5,6,7,8,9,10,\dots\}$

$\mathbb{Z}^+ = \{1,2,3,4,5,6,7,8,9,10,\dots\}$

$\mathbb{Z} = \{\dots,-10,-9,-8,-7,-6,-5,-4,-3,-2,-1,1,2,3,4,5,6,7,8,9,10,\dots\}$

n_{ek} = The k th element of an even number in the set of natural or integer numbers

n_{ok} = The k th element of an odd number in the set of natural or integer numbers

n_{xk} = The k th element of an odd or even number in the set of natural or integer numbers,

belonging to the closed interval $[1,2, \dots, n_{xk}]$

n_q = The number of even and odd numbers in the closed interval $[1,2, \dots, n_{xk}]$ after

selecting the desired n_{xk} for the Collatz conjecture

Some examples of the Collatz conjecture sequence for $n \in \mathbb{N}$ are presented in the

Following ;

$$\text{Scc (1)} = \{1,4,2,1\}$$

$$\text{Scc (2)} = \{2,1\} *$$

$$\text{Scc (3)} = \{3,10,5,16,8,4,2,1\}$$

$$\text{Scc (4)} = \{4,2,1\}$$

$$\text{Scc (5)} = \{5,16,8,4,2,1\}$$

$$\text{Scc (6)} = \{6,3,10,5,16,8,4,2,1\}$$

$$\text{Scc (7)} = \{7,22,11,17,52,26,13,40,20,10,5,16,8,4,2,1\}$$

$$\text{Scc (8)} = \{8,4,2,1\}$$

$$\text{Scc (9)} = \{9,28,14,7,22,11,17,52,26,13,40,20,10,5,16,8,4,2,1\}$$

$$\text{Scc (10)} = \{10,5,16,8,4,2,1\}$$

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If we continue to repeat the rules of the Collatz conjecture after reaching 1 in the above sequence members, we will again reach 1.

An interesting critical point to note is that by carefully observing each of the above examples related to $\text{Scc}(n)$, two consecutive divisions are found with even quotients. Accordingly, the process of reaching 1 becomes faster in the Collatz conjecture and we become very closer to the desired final number. It is useful to compare the above Collatz conjecture sequences, $\text{Scc}(n)$, with factor 1 instead of 3.

(A) The original Collatz conjecture rules are as follows:

$$f(n) = \begin{cases} n / 2 & ; \text{if } n \text{ is even} \\ (3n + 1) / 2 & ; \text{if } n \text{ is odd} \end{cases}$$

($n / 2$ and $(3n + 1) / 2$ are even numbers)

(B) The rules of the Collatz conjecture for factor 1 are as follows:

$$f(n) = \begin{cases} n_{ek} / 2 \\ n_{ok} + 1 / 2 \end{cases}$$

$$S_{cc} (1) = \{1,2,1\}$$

$$S_{cc} (2) = \{2,1\}^*$$

$$S_{cc} (3) = \{3,4,2,1\}$$

$$S_{cc} (4) = \{4,2,1\}$$

$$S_{cc} (5) = \{5,6,3,4,2,1\}$$

$$S_{cc} (6) = \{6,3,4,2,1\}$$

$$S_{cc} (7) = \{7,8,4,2,1\}$$

$$\text{Scc (8)} = \{8,4,2,1\}$$

$$\text{Scc (9)} = \{9,10,5,6,3,4,2,1\}$$

$$\text{Scc (10)} = \{10,5,6,3,4,2,1\}$$

$$\text{Scc (11)} = \{11,12,6,3,4,2,1\}$$

$$\text{Scc (12)} = \{12,6,3,4,2,1\}$$

$$\text{Scc (13)} = \{13,14,7,8,4,2,1\}$$

$$\text{Scc (14)} = \{14,7,8,4,2,1\}$$

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The proof

Numbers obtained from the Collatz conjecture formula are, in fact, a descending sequence of integer or natural numbers, and they do not need to be positive or multiplied by 3 if they are odd; that is, for $n \neq 0$, $n \in \mathbb{N} \vee \mathbb{Z}^+$.

Instead of 3, if you choose an odd number, you can also multiply the number by 1 or any other odd numbers. In this case, the result will be the same and will always reach 1 after the successive division by 2. The only differences are the delay in reaching 1 and the increased or decreased number of operations and processes in the Collatz conjecture formula. The proof of the following theorems and lemmas can be found in articles and books on the set and number theory (Dixon et al., 2010; Grossman, 2010; Neely, 2020; Pommersheim et al., 2010).

Propositions

P (theorem): The multiplication of the two odd numbers (\mathbb{N} and Z^+) results in an odd number (Grossman, 2010).

Q (theorem): Each even number (\mathbb{N} or Z^+) is divisible by 2 (Grossman, 2010).

R (theorem): The sum of two odd numbers is an even number; the sum of two odd numbers (\mathbb{N} or Z^+) and 1 is an even number (Grossman, 2010).

Thus, we can deduce that:

$(P \wedge R) \Rightarrow (3n_{ok})$ is an odd number, and $(3n_{ok} + 1)$ is an even number.

S: Each set of numbers used in the Collatz conjecture formula, (\mathbb{N} or Z^+), is a countable set, and its resultant sequences are subsets of the number sets \mathbb{N} or Z^+ , which are finite sets (Neely, 2020).

T: We can write

$\forall m, k \in \mathbb{N}, n \in \mathbb{N} \vee Z^+, n, k \neq 0$

$n_1 / 2 = n_2, n_2 < n_1 \Rightarrow n_2 / 2 = n_3, n_3 < n_2 \Rightarrow \dots, n_k / 2 = n_{k+1}, n_{k+1} < n_k \Rightarrow$

$S_{cc}(n_k) = \{(n_1 / 2), ((n_2 / 2) / 2), (((n_2 / 2) / 2) / 2), \dots, (n_k / 2^m), \dots, 1\}$

U: It is obvious that after the successive division by 2 in the Collatz conjecture

sequences, $S_{cc}(n)$, we always reach the same number because of the unique result of

the division (a theorem in number theory (Dixon et al., 2010; Pommersheim et al.,

2010) and the recursive nature of the Collatz conjecture sequences, $S_{cc}(n)$, that is;

$$S_{cc}(1) = \{1, S_{cc}(4)\}$$

$$S_{cc}(2) = \{2,1\} *$$

$$S_{cc}(3) = \{3, S_{cc}(10)\}$$

$$S_{cc}(4) = \{4, S_{cc}(2)\}$$

$$S_{cc}(5) = \{5,16,8, S_{cc}(4)\}$$

$$S_{cc}(6) = \{6, S_{cc}(3)\}$$

$$S_{cc}(7) = \{7,22, S_{cc}(11)\}$$

$$S_{cc}(8) = \{8, S_{cc}(4)\}$$

$$S_{cc}(9) = \{9,28,14, S_{cc}(7)\}$$

$$S_{cc}(10) = \{10, S_{cc}(5)\}$$

$$S_{cc}(11) = \{11,34,17,52,26,13,40,20,10, S_{cc}(5)\}$$

$$S_{cc}(12) = \{12, S_{cc}(6)\}$$

$$S_{cc}(13) = \{13,40,20, S_{cc}(10)\}$$

$$S_{cc}(14) = \{14, S_{cc}(7)\}$$

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V: The speed and rate of reaching 1 through the successive division by 2 in the Collatz conjecture sequences, $S_{cc}(n)$, are higher than the delay that occurs by $3n_{ok} + 1$.

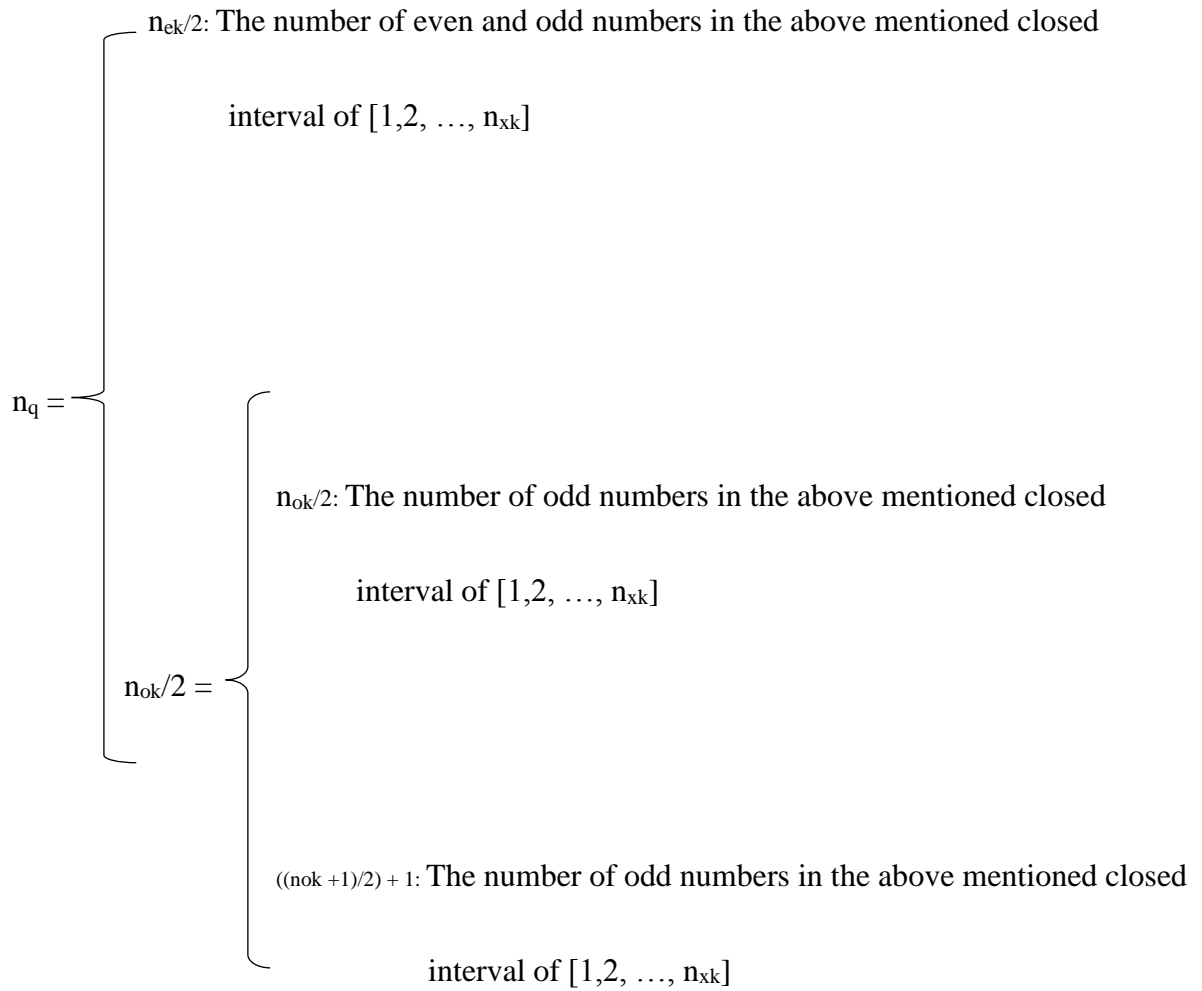
W: In the Collatz conjecture, the numerical coefficient 3 is the smallest odd number of greater than 1, causing a delay in reaching 1 that is not too long.

X: The number of even and odd numbers in the closed interval of $[1, 2, \dots, n_{xk}] \wedge \forall n_{xk} \in$

$\mathbb{N} \vee \mathbb{Z}^+$, $x, k \in \mathbb{N}$, $x, n, k \neq 0$, where n_{ek} is even and n_{ok} is odd, is calculated as

follows:

Here, n_q is the number of even and odd numbers in the above mentioned closed interval of ; $[1, 2, \dots, n_{xk}]$ after the selection of the desired n_{xk} for the Collatz conjecture.



Due to the above, we must find a relationship between the even and odd numbers of the selected or desired n_{xk} , which is obtained by the number of consecutive divisions by 2 to reach 1 in the Collatz conjecture sequence, $Sc(n)$, because of the following facts.

[1] In each of the Collatz conjecture sequences, it can be observed that the number of consecutive divisions necessary to reach the final and desired number 1 is equal to the number of even numbers existing in that particular Collatz conjecture sequence, $Sc(n)$.

[2] Therefore, by finding the number of even numbers in each Collatz conjecture sequence, $Sc(n)$, in the closed interval $[1, 2, \dots, n_{xk}]$, we can certainly determine the number of consecutive divisions necessary to reach 1 in the Collatz conjecture. However, finding the above mentioned number is very complicated because of the operation of $(3n_{ok} + 1)$, with n_{ok} being odd, that repeatedly changes numbers and also changes the order and position of numbers in the sets of \mathbb{N} or \mathbb{Z}^+ as well as their evenness or oddness.

All examples provided in Table 1 indicate that the members of the Collatz conjecture sequences, $Sc(n)$, easily show the number of divisions by 2 required to reach the desired number 1 in the Collatz conjecture, $Sc(n)$, which is equal to the number of even numbers in that Collatz conjecture sequence, $Sc(n)$, i.e.;

Table 1. The characteristics of the Scc(n) members.

the Collatz conjecture sequence , Scc(n)	Number of even numbers in Scc(n)	Number of odd numbers in Scc(n)	Number of consecutive division operations required in the Collatz conjecture to reach the final and desired number 1	$n / 2$ if n is even ($n_{ek} / 2$)	$(3n+ 1)$ if n is odd ($n_{ok} / 2$)	Number of times that $(3n_{ok}+ 1)$ is used
Scc(1)	2	1	2	----	2	1
Scc(2)	1	0	1	1	---	0
Scc(3)	5	2	5	---	5	2
Scc(4)	2	0	2	2	---	0
Scc(5)	4	1	4	---	8	1
Scc(6)	6	2	6	3	---	2
Scc(7)	11	5	11	---	11	5
Scc(8)	3	0	3	4	---	0
Scc(9)	13	6	13	---	14	6
Scc(10)	5	1	5	5	---	1
Scc(11)	10	4	10	---	17	4
Scc(12)	7	2	7	6	---	2
Scc(13)	7	2	7	---	20	2
Scc(14)	2	5	2	7	---	5
...

Y: To reach 1 in the Collatz conjecture, we need to divide even numbers for m times

by 2, and then, we can write;

$$\forall n_{xk} \in n_{ek} \vee (3n_{ok} + 1) \ \& \ (k, n_{xk}, x, n_{ek}, n_{ok} \in \mathbb{N} \vee \mathbb{Z}^+, n_{xk}, x, n_{ek}, n_{ok}, k \neq 0),$$

n_{xk} = The k th element of an odd or even number in the set $\mathbb{N} \vee \mathbb{Z}^+$ that belongs to the

closed interval $[1, 2, \dots, n_{xk}]$

n_{ek} = The k th element of an even number in the set $\mathbb{N} \vee \mathbb{Z}$

n_{ok} = The k th element of an odd number in the set $\mathbb{N} \vee \mathbb{Z}^+$

the Collatz conjecture rules, $f(n)$, is;

$$f(n) = \begin{cases} n / 2 ; \text{ if } n \text{ is even} \\ (3n + 1) / 2 ; \text{ if } n \text{ is odd} \end{cases}$$

($n / 2$ and $(3n + 1) / 2$ are even numbers)

So, we can assume that;

$$n_{ek} / 2 \approx (3 n_{ok} + 1) / 2 \approx n_{xk} / 2$$

then we can write;

$$f(n) = (n_{xk} / 2^m)$$

Here, $f(n)$ is the relation in the rules of the Collatz conjecture.

Thus, to reach 1, after which the successive division by 2 is not need, the following limit must be equal to 1:

$$\lim_{m \rightarrow 0} f(n) = \lim_{m \rightarrow 0} [(n_{xk} / 2^m) = 1] = [n_{xk} = 1], \text{ that is ;}$$

After m times division by 2, we will definitely reach 1.

Also, for; $f(n) = (n_{xk} / 2^m) = 1$, there should be; $n_{xk} = 2$. So, if we rewrite the $f(n)$ for the last division as; $f(n) = n_{xk} / 2 = 1$, and then by multiplying the both sides of $f(n)$ by 2, we will have; $2 \cdot (n_{xk} / 2) = 2 \cdot (1) \Rightarrow n_{xk} = 2$, And again, we can see that; After m times division by 2, we will definitely reach 1.

Now, based on all the above propositions and particularly by mentioning the proposition S, each set of numbers used in the Collatz conjecture formula, \mathbb{N} or \mathbb{Z}^+ , is countable and its resultant sequences are subsets of the number sets \mathbb{N} or \mathbb{Z}^+ , which are finite sets (refs. 2, 3,4).

$$(P \wedge Q \wedge R \wedge S \wedge T \wedge U \wedge V \wedge W \wedge X \wedge Y) \Rightarrow Z \text{ (Q.E.D/■)}$$

It can be concluded with certainty that the Collatz sequence eventually reaches 1 for all initial arbitrary values of \mathbb{N} or \mathbb{Z}^+ . Now, the proof of the Collatz conjecture is completed, (Q.E.D/■).

The Mechanism of the Collatz conjecture relation and Conclusion

Now, I would like to introduce a new relation in mathematics that I called it;

Self _Shrunken Relation = $\mathbb{R} \ominus (x)$.

And I define it as a relation that, by its inherent definition, is "recursive".

It repeatedly and successively until reaching a fixed number according to a specific definition for it, dividing by a real number \mathbb{R} , gets small and smaller.

And the Collatz conjecture relation can be considered as an example of a

Self _Shrunken Relation = $\mathbb{R} \ominus (x)$.

More precisely, due to the definition of the Collatz conjecture relation; if the starting or initial number n , is even after dividing by 2 will get smaller or if n is odd, although multiplied by 3, but, after adding with 1, it becomes an even number and therefore is divisible by 2, which becomes smaller.

Those processes are repeated and for each time, with fluctuations and jumps similar to the damping oscillations of a spring, after a slow or fast process, reaches a maximum value (identical to probability distribution diagrams) from which, due to the consecutive divisions by 2 that mentioned above, it will finally and definitely reach the number 2 and then the desired and final number 1; that usually occurs after the two consecutive and immediate divisions by 2.

References

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