TECHNIUM

Technium Vol. 8, pp.12-24 (2023)
ISSN: 2668-778X
www.techniumscience.com

Application of Laplace Transform in Solving Linear Differential Equations with Constant Coefficients

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Abstract

In recent years, the interest in using Laplace transforms as a useful method to solve certain types of differential equations and integral equations has grown significantly. In addition, the applications of Laplace transform are closely related to some important parts of pure mathematics. Laplace transform is one of the methods for solving differential equations. This method is especially useful for solving inhomogeneous differential equations with constant coefficients and it has advantages compared to other methods of solving differential equations. Linear differential equations with constant coefficients are among the equations that can be solved using the Laplace transform. Because the transformation Laplace is one of the transformations that easily converts exponential functions, trigonometric functions, and logarithmic functions into algebraic functions. Therefore, it is considered a better method for solving linear differential equations with constant coefficients.

Keywords: Linear Differential Equations, Laplace Transform, Integral Transforms, Virtual Integrals

1. INTRODUCTION

One of the most useful tools for solving linear differential equations is integral transformations. Integral transformations are relational in the form

$$F(s)=\int_{\alpha}^{\beta} K(s,t)\, f(t)\,dt \qquad (1)$$

which transforms the assumed function f into another function F by integration. The function F is called transformed or transformed f and K is called the transformation kernel. Its general plan is to convert the problem related to f into a simpler problem related to F by using equation (1) and solve this simpler problem and then obtain the desired function f by using its converter, i.e.F. By choosing an appropriate kernel K and the limits of integration and, we can mostly simplify the problem involved in a linear differential equation fundamentally. Some integral transformations have many applications; and each of them is suitable for the appropriate types of issues [1].

In this article, research has been done on the application of Laplace transform, which is a type of integral transformation, in solving linear differential equations. Laplace transform has a special place in solving differential equations with initial conditions. In this transformation, the appropriate function f of the transform t is transformed into a function like F of the transform s, and the function is transferred from the space t to the space s. In fact, by using the Laplace transform, a linear differential equation (including a function of the transform t) with the initial conditions becomes an algebraic equation in terms of s becomes [2].

2. Laplace transform

Definition 1. Suppose the function f(t) is defined in the interval $(0,+\infty)$. If the function F(s) is as:

$$F(s)=\int_{0}^{+\infty} e^{-st} f(t)dt$$

is considered, then the function F(s) is called the Laplace transform of the function f(t). Provided that the above integral exists. We denote Laplace transform f by F or L(f) and also F(s) by L(f(t)) issues [3].

The function f(t) is called the inverse of the transformation F(s) and it is represented by the symbol $L^{-1}\{F\}$

$$f(t)=L^{-1}\{F\}$$

We always show the Laplace transform of the functions with the corresponding capital letters, that is, the Laplace transform of the function g(t) with G(s) and the Laplace transform of the function h(t) with H(s) and...,.

3. Conditions for existence of the Laplace transform of a function

Before stating the existence theorem of Laplace transform of a function, it is necessary to state two titles of piecewise continuous and function of exponential order [4]. Piecewise continuous:

Definition 2. The function f(t) on the interval (a,b) is called piecewise continuous if we can divide this interval into smaller intervals such as (c,d) so that:

a) f(t) is continuous on any open interval (c,d).

b) f(t) has a certain limit at the endpoints of the intervals, that is, $\lim_{t\to d^-} f(t)$ and $\lim_{t\to c^+} f(t)$ exist and are certain.

Definition 3. The function f(t) is said to be piecewise continuous on the interval [0.∞], if it is on the interval [0.∞] for every $N>0$ piecewise continuous [5].

4. Function of exponential order

Definition 4. The function f(t) on the distance $[0,\infty)$ is said to be of exponential order if there are positive constants M and b such that:

$$|f(t)| < Me^{bt} \quad , \quad t \geq t_0 \qquad (1)$$

If the emphasis is on b, we say f(t) is of exponential order e^{bt} , when $t \to \infty$, we write

$$f(t) = o(e^{bt}) \qquad (2)$$

When we say that f(t) is of exponential order e^{bt}, it means that for the values of b, the function f(t) applies in the conditions of definition (1), that is, the growth of f(t) is less than the growth of Me^{bt}. For example, the function e^{t^2} is not of exponential order because for each b we have:

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{bt}} = \lim_{t\to\infty} e^{t(t-b)} = +\infty$$

5. Existence of the Laplace transform of a function

Theorem 1. If the function f(t) has the following conditions:
a) is piecewise continuous in any finite distance $0 \leq t \leq T$.

b) for $t > T$ is the co-exponential of e^{at}.
Then the Laplace transform of the function f(t) exists for all $s > a$.

Proof. $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt$

Because the function f(t) is piecewise continuous in the interval $0 \leq t \leq T$, therefore $e^{-st} f(t)$ is also piecewise continuous in the interval $0 \leq t \leq T$ and $\int_0^T e^{-st} f(t)dt$ will exist.

$$|L\{f(t)\}| = \left|\int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt\right| \leq \left|\int_0^T e^{-st} f(t)dt\right| + \left|\int_T^\infty e^{-st} f(t)dt\right|$$

Now we show that $\int_T^\infty e^{-st} f(t)dt$ also exists

$$\left|\int_T^\infty e^{-st} f(t)dt\right| \leq \int_T^\infty e^{-st} |f(t)|dt \leq \int_0^\infty e^{-st} |f(t)|\, dt$$

$$\leq M\int_0^\infty e^{-st} e^{at} dt = \frac{M}{s-a} \quad , \quad s > a$$

Note: The conditions of the above case are sufficient conditions and not necessary. That is, there are functions that do not apply to the existence theorem of the Laplace transform of a function, but have the Laplace transform [6].

Theorem 2. If the function f(t) applies under the condition of Laplace transform existence theorem, then

$$\lim_{s\to\infty} F(s) = 0$$

That is, if $\lim_{s\to\infty} F(s) \neq 0$,then f(t) cannot have the conditions of Laplace transform existence theorem [6].

6. Properties of the Laplace transform

Theorem 3. (Laplace transform linearity). Suppose $f_1(t)$ and $f_2(t)$ have Laplace transform, then [6].

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$

Proof. We consider the definition of Laplace transform:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t)\right)dt).$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty e^{-st} f_2(t)dt$$

$$= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$

7. Laplace transform of derivatives

Theorem 4. Suppose that the real function f(t) is continuous in the interval $[0, \infty)$ and is of exponential order e^{at} and also suppose that $f'(t)$ is piecewise continuous in any finite closed interval $0 \le t \le b$. Then $L\{f'\}$ exists for every $s > a$ and we have:

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Theorem 5. Suppose that the real function f(t) has $(n-1)$ a continuous derivative for $t \ge 0$. In other words, functions $f, f', f'', \cdots, f^{(n-2)}$ are continuous in the interval $[0, \infty)$. Also, all the above derivatives are of exponential order e^{at} and the nthderivative is f or $f^{(n)}(t)$ in any closed interval $0 \le t \le b$ is piecewise continuous. Then $L\left\{f^{(n)}\right\}$ exists for every $s > a$ and we have:

$$L\left\{f^{(n)}(t)\right\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$$

Example 1. Prove that $L\left\{t^n\right\} = \dfrac{n!}{s^{n+1}} \qquad s > 0$

Solve. Since it is $f(t) = t^n$, then we have:

$$\begin{aligned}
&f(t)=t^n \quad , \qquad f(0)=0\\
&f'(t)=nt^{n-1} \; , \qquad f'(0)=0\\
&\vdots\\
&f^{(n-1)}(t)=n!t \quad , \quad f^{(n-1)}(0)=0\\
&f^{(n)}(t)=n!\\
&L\left\{f^{(n)}(t)\right\}=L\{n!\}=\frac{n!}{s}=s^nL\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)\\
&=s^nL\{f(t)\}\Rightarrow L\{f(t)\}=\frac{n!}{s^{n+1}}
\end{aligned}$$

[7].

8. Laplace transform integral

If $L\{f(t)\}=F(s)$ with condition $s>0$ then we have:

$$L\left\{\int_0^t f(t)dt\right\}=\frac{1}{s}F(s)$$

$$L\left\{\int_0^t\int_0^t f(t)dtdt\right\}=\frac{1}{s^2}F(s)$$

And in the inverse form, if $L^{-1}\{F(s)\}=f(t)$, then we have:

$$L^{-1}\left\{\frac{1}{s}F(s)\right\}=\int_0^t f(t)dt$$

$$L^{-1}\left\{\frac{1}{s^2}F(s)\right\}=\int_0^t\int_0^t f(t)dtdt$$

[8].

Example 2. Get the integral $\int_0^t \sin 4xdx$ Laplace transform [9].

We use the integral Laplace transform formula**:**

$$L\left\{\int_0^t f(t)dt\right\}=\frac{1}{s}F(s)$$

$$L\left\{\int_0^t \sin 4tdt\right\}=\frac{1}{s}L\{\sin 4t\} \qquad a=4$$

According to the Laplace transform formula $\sin at$, that is, $L\{\sin at\}=\frac{a}{s^2+a^2}$ is equal to:

$$=\frac{1}{s}\frac{4}{s^2+16}$$

Theorem 6. Suppose that the Laplace transform of the function f(t) exists for $s>\alpha$ and we have:

$$L\{f(t)\}=F(s)$$

Then for each $(a\in\square),s>\alpha+a$ we have: $L\left\{e^{at}f(t)\right\}=F(s-a)$

The proof of the above property is as follows $F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)dt=\int_0^\infty e^{-st}[e^{at}f(t)]dt=L\left\{e^{at}f(t)\right\}$

Theorem 7. Suppose that the function f(t) has all the conditions in the definition of the Laplace transform. Also suppose that F is the Laplace transform of the given function such that:

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

Then we have: $$L\left\{t^n f(t)\right\}=(-1)^n\frac{d^n}{ds^n}L\left\{f(t)\right\}=(-1)^n\frac{d^n}{ds^n}[F(s)]$$

9. Laplace transform of some elementary functions

Now we will determine the Laplace transform of some preliminary functions:

Example 3. Get the Laplace transform $f(t)=1$ for $t>0$.

Solve.

$$L\{1\}=\int_0^\infty e^{-st}(1)dt=\lim_{R\to\infty}\int_0^R e^{-st}dt=\lim_{R\to\infty}\left[\frac{-e^{-st}}{s}\right]_0^R$$

$$=\lim_{R\to\infty}\left[\frac{1}{s}-\frac{e^{-sR}}{s}\right]=\frac{1}{s}-0=\frac{1}{s}\quad,\quad s>0$$

Example 4. It is desirable to solve $L\left\{e^{ax}\right\}$

Solve.

$$L\left\{e^{at}\right\}=\int_0^\infty e^{-st}e^{at}dx=\int_0^\infty e^{-(s-a)t}dt=\left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty$$

$$=0+\frac{1}{s-a}\quad,\quad s>0$$

In this example, if it is $s\leq a$, then the power of e will be positive and the integral will diverge because we want the integral to converge, so we assume that it is $s>a$.

Example 5. It is desirable to determine (a) $L\{\sin at\}$, (b) $L\{\cos at\}$

Solve. Using the previous example we have: $L\left\{e^{iat}\right\}=\frac{1}{s-ai}$, because $\frac{1}{s-ai}=\frac{s+ai}{s^2+a^2}$ using Euler's formula we have $e^{iat}=\cos at+i\sin at$

So

$$L\left\{e^{iat}\right\}=L\{\cos at+i\sin at\}$$

Based on the linear property of the Laplace transform, we can write:

$$=L\{\cos at\}+iL\{\sin at\}=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}$$

So we have: $L\{\cos at\}=\frac{s}{s^2+a^2}$ and $L\{\sin at\}=\frac{a}{s^2+a^2}$ here is $s>0$.

Example 6. It is desirable to solve:

It is desirable to solve (a) $L\{\sinh at\}$, (b) $L\{\cosh at\}$.

Solve. Because $L\{e^{at}\}=\dfrac{1}{s-a}$ and $L\{e^{-at}\}=\dfrac{1}{s+a}$ then

$$L\{\sinh at\}=L\left\{\frac{e^{at}-e^{-at}}{2}\right\}=\frac{1}{2}\left[L\{e^{at}\}-L\{e^{-at}\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]=\frac{a}{s^2-a^2}$$

With the same method and considering that $\cosh at=\dfrac{e^{at}+e^{-at}}{2}$ we have $L\{\cosh at\}=\dfrac{s}{s^2-a^2}$

Example 7. Get the Laplace transform of the unit step function below.

$$u_c(t)=\begin{cases}0 & 0\le t<c\\ 1 & t\ge c\end{cases}$$

Solve.

$$L\left(u_c(t)\right)=\int_0^\infty e^{-st}u_c(t)dt=\int_0^c e^{-st}.0dt+\int_c^\infty e^{-st}.1dt=\lim_{b\to\infty}\int_c^b e^{-st}dt$$

$$=\lim_{b\to\infty}(-\frac{1}{s}e^{-st})\Big|_c^b=\lim_{b\to\infty}(-\frac{1}{s}e^{-sb}+\frac{1}{s}e^{-cs})=\frac{e^{-cs}}{s}$$

with condition $s>0$ [10].

Table (1) Laplace transform formulas of some functions

No	$f(t)$	The Laplace transform of the function f(t) means $F(s)=L\{f(t)\}$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	e^{at}	$\frac{1}{s-a}$
4	$\sin at$	$\frac{a}{s^2+a^2}$
5	$\cos at$	$\frac{s}{s^2+a^2}$
6	$t^n \quad (n\in \square)$	$\frac{n!}{s^{n+1}}$
7	$t^n e^{at} \quad (n\in \square)$	$\frac{n!}{(s-a)^{n+1}}$
8	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
9	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

10	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$
11	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
12	$\sinh at$	$\frac{a}{s^2-a^2}$
13	$\cosh at$	$\frac{s}{s^2-a^2}$
14	$t^n f(t) \qquad (n\in\mathbb{N})$	$(-1)^n F^{(n)}(s)$
15	$u_a(t)$	$\frac{e^{-as}}{s}$

[10].

10. The inverse of the Laplace transform

If $L\{f(t)\}=F(s)$, then we call f(t) the inverse of Laplace transform F(s) and write: $L^{-1}\{F(s)\}=f(t)$

It can be shown that the inverse of the Laplace transform is also a linear transformation (Ardbili, 2014).

Theorem 8. The inverse of the Laplace transform is linear.

Suppose $L\{f_1(t)\}=F_1(s)$ and $L\{f_2(t)\}=F_2(s)$ then:

$$L^{-1}\{c_1F_1(s)+c_2F_2(s)\}=c_1L^{-1}\{F_1(s)\}+c_2L^{-1}\{F_2(s)\}$$

Proof. Therefore, we have the Laplace transform property that:

$$L\{c_1 f_1(t)+c_2 f_2(t)\}=c_1F_1(s)+c_2F_2(s)$$

Thus, we have the inverse definition of the Laplace transform:

$$L^{-1}\{c_1F_1(s)+c_2F_2(s)\}=c_1 f_1(t)+c_2 f_2(t)$$

On the other hand $f_1(t)=L^{-1}\{F_1(s)\}$, $f_2(t)=L^{-1}\{F_2(s)\}$

So $L^{-1}\{c_1F_1(s)+c_2F_2(s)\}=c_1L^{-1}\{F_1(s)\}+c_2L^{-1}\{F_2(s)\}$

Table (2) inverse of Laplace transform of some functions

$F(s)$	$L^{-1}\{F(s)\} = f(t)$
$\frac{1}{s}$	1
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$

[11].

Example 8. Get the inverse of the Laplace transform of the expression $\frac{7}{s+3}-\frac{6}{s^2+4}$.

Solve.

$$L^{-1}\left\{\frac{7}{s+3}-\frac{6}{s^2+4}\right\}$$

$$=L^{-1}\left\{\frac{7}{s+3}\right\}-L^{-1}\left\{\frac{6}{s^2+4}\right\}=7L^{-1}\left\{\frac{1}{s+3}\right\}-3L^{-1}\left\{\frac{2}{s^2+4}\right\}=7e^{-3t}-3\sin 2t$$

We used the following formulas in the above question.

$$L^{-1}\left\{\frac{1}{s-a}\right\}=e^{at} \quad , \; L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$$

Example 9. Get the inverse transformation of the function $F(s)=\frac{e^{-2s}}{s^2+s-2}$ [5].

Solve. We analyze the denominator of the fraction as $s^2+s-2=(s+2)(s-1)$. Then we write the fraction $\frac{1}{s^2+s-2}=\frac{1}{(s+2)(s-1)}$ as the sum of its partial fractions:

$$\frac{1}{s^2+s-2}=\frac{1}{(s+2)(s-1)}=\frac{A}{s+2}+\frac{B}{s-1}=\frac{(A+B)s+2B-A}{(s+2)(s-1)}$$

$$A+B=0$$

$$-A+2B=1$$

From the solution of this system, it is obtained that $B=\frac{1}{3}$ and $A=-\frac{1}{3}$, so the given function can be written as:

$$F(s)=-\frac{1}{3}\frac{e^{-2s}}{s+2}+\frac{1}{3}\frac{e^{-2s}}{s-1}$$

According to the formulas $L^{-1}\left\{\frac{1}{s+2}\right\}=e^{-2t}$ and $L^{-1}\left\{\frac{1}{s-1}\right\}=e^{t}$, we will have:

$$f(t)=-\frac{1}{3}e^{-2(t-2)}u_2(t)+\frac{1}{3}e^{t-2}u_2(t)$$

Or

$$f(t)=\frac{1}{3}[e^{t-2}-e^{-2(t-2)}]u_2(t)$$

[3].

11. Application of Laplace transform in solving linear differential equations with constant coefficients

The main subject of this article is research on the use of Laplace transform in solving linear differential equations with constant coefficients and initial conditions. These equations are in general form.

$$a_n y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1 y'+a_0 y=b$$

with initial conditions such as are:

$$y(0)=c_0\,,\,y'(0)=c_1\,,\cdots,\,y^{(n-1)}(0)=c_{n-1}$$

The basic theorem that by using the Laplace transform can be used to solve linear differential equations of the first order nth with constant coefficients, the Laplace transform theorem is the derivative of a function, which we rewrite below.

The case Suppose that the real function f(t) has $(n-1)$ a continuous derivative for $t\geq 0$. In other words, functions $f, f', f'',\cdots, f^{(n-2)}$ are continuous in the interval $[0,\infty)$. Also, all the above derivatives are of exponential order e^{at} and the nth derivative of f or $f^{(n)}(t)$ is piecewise continuous in any closed interval $0\leq t\leq b$. Then $L\left\{f^{(n)}\right\}$ exists for every $s>a$ and we have:

$$L\left\{f^{(n)}(t)\right\}=s^n L\left\{f(t)\right\}-s^{n-1}f(0)-s^{n-2}f'(0)$$
$$-s^{n-3}f''(0)-\cdots-f^{(n-1)}(0)$$

Since every solution is a linear differential equation with constant coefficients and its derivatives of exponential order, and as a result, it must have the Laplace transform; therefore, the above theorem can be used in solving linear differential equations with constant coefficients.

To solve the above-mentioned equations, we proceed as follows:

1. We calculate the Laplace transform of the sides of the differential equation, from which the algebraic equation in terms of s and F(s) is obtained.
2. We replace the initial conditions and solve the obtained algebraic equation in terms of F(s).
3. We calculate the inverse of the Laplace transform of the sides of equation 2 to obtain the function y(t).

Example 10. Solve the differential equation using Laplace transforms. $\frac{dy}{dt}-2y=e^{5t}$, $y(0)=3$

Solve. 1- First, we calculate the Laplace transform of the sides of the equation

$$L\left\{\frac{dy}{dt}\right\}-2L\{y(t)\}=L\{e^{5t}\}$$

Because it is $L\left\{\frac{dy}{dt}\right\}=sF(s)-y(0)$ and $L\{e^{5t}\}=\frac{1}{s-5}$, so we have: $sF(s)-y(0)-2F(s)=\frac{1}{s-5}$

The initial conditions are replaced and the resulting algebraic equation is solved in terms of F(s):

$$(s-2)F(s)-3=\frac{1}{s-5} \qquad y(0)=3$$

Because:

$$(s-2)F(s)=\frac{1}{s-5}+3\Rightarrow(s-2)F(s)=\frac{1+3(s-5)}{s-5}\Rightarrow F(s)=\frac{1+3(s-5)}{(s-2)(s-5)}$$

$$(s-2)F(s)=\frac{1}{s-5}+3\Rightarrow(s-2)F(s)=\frac{1+3(s-5)}{s-5}\Rightarrow F(s)=\frac{1+3(s-5)}{(s-2)(s-5)}$$

$$F(s)=\frac{3s-14}{(s-2)(s-5)}$$

In order to calculate the inverse of the Laplace transform of the function F(s), we first decompose the right side of the last relation into its partial fractions.

$$\frac{3s-14}{(s-2)(s-5)}=\frac{A}{s-2}-\frac{B}{s-5}$$

$$3s-14=A(s-5)-B(s-2)$$

It is calculated as follows by replacing the values of A and B in the above equation:

$$6-14=A(2-5)\Rightarrow A=\frac{8}{3}$$

$$15-14=B(5-2)\Rightarrow B=\frac{1}{3}$$

The inverse solution of the Laplace transform is calculated as follows:

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{3s-14}{(s-2)(s-5)}\right\}$$

$$L^{-1}\{F(s)\}=\frac{8}{3}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{s-5}\right\}$$

$$y(t)=\frac{8}{3}e^{2t}+\frac{1}{3}e^{5t}$$

Example 11. obtain the solution of the differential equation $f''(t)+8f'(t)+16f(t)=0$ considering the initial conditions $f(0)=2$ and $f'(0)=1$ by Laplace transform.

Solve. Using the Laplace transform method, we have:

$$L\{f''(t)\}+8L\{f'(t)\}+16L\{f(t)\}=0 \qquad (1)$$

Using the Laplace transform formula, the derivative of the nth order of the function

$$L\{f^{(n)}(t)\}=s^nL\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)$$

$$L\{f''(t)\}=s^2L\{f(t)\}-s^{2-1}f(0)-s^{2-2}f'(0)=s^2F(s)-s2-s^01=s^2F(s)-1$$

$$8L\{f'(t)\}=8\left[sL\{f(t)\}-s^{1-1}2\right]=8sF(s)-16$$

We set the prices in relation (1):

$$(s^2F(s)-2s-1)+(8sF(s)-16)+16F(s)=0$$

After simplifying, it is obtained that:

$$F(s)=\frac{2s+17}{s^2+8s+16}=\frac{2(s+4)+9}{(s+4)^2}\Rightarrow f(t)=L^{-1}\left\{\frac{2(s+4)+9}{(s+4)^2}\right\}$$

$$\Rightarrow f(t)=e^{-4t}L^{-1}\left\{\frac{2s+9}{s^2}\right\}=e^{-4t}(2+9t)$$

[7].

Example 12. Solve the following differential equation using Laplace transforms.

$$\frac{d^2y}{dt^2}-2\frac{dy}{dt}-8y=0 \qquad y(0)=3 \qquad y'(0)=6$$

Solve. First, we calculate the Laplace transform of the sides of the equation.

$$L\left\{\frac{d^2y}{dt^2}\right\}-2L\left\{\frac{dy}{dt}\right\}-8L\{y\}=L\{0\}$$

$$[s^2F(s)-sy(0)-y'(0)]-2[sF(s)-y(0)]-8F(s)=0$$

We replace the initial conditions and solve the resulting algebraic equation in terms of F(s).

$$s^2F(s)-3s-6-2sF(s)+6-8F(s)=0 \qquad (y(0)=3\ ,\ y'(0)=6)$$

$$[s^2-2s-8]F(s)-3s=0$$

$$F(s)=\frac{3s}{s^2-2s-8}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{3s}{s^2-2s-8}\right\}\Rightarrow L^{-1}\{F(s)\}=L^{-1}\left\{\frac{3s}{(s-4)(s+2)}\right\}$$

$$L^{-1}\{F(s)\}=2L^{-1}\left\{\frac{1}{s-4}\right\}+L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$y(t)=2e^{4t}+e^{-2t}$$

[11].

12. CONCLUSION

From the study about the application of Laplace transform in solving linear differential equations with constant coefficients, we came to the conclusion that Laplace transform is a very easy and accurate method to achieve the solution of linear differential equations with constant coefficients. Just as with the help of Taylor's series and McLaurin’s series, it is possible to convert functions that have a closed and complex form into functions in the form of algebraic expressions and perform the desired operations on them, in the same way, with the help of Laplace transform, the function can be transformed by an integral transformation called Laplace transformed from one form to another. Its main difference is that Taylor's series and McLaurin’s series transform the function from the genus of a variable back into a function whose variable is the same as the first variable. While in the Laplace transform, it changes completely as a result of the transformation.

As well as the operation is performed on top of the functions, sometimes we face problems regarding the structure of the function. In this case, the Laplace transform helps to convert the function into a simpler form and later perform the desired operations on it. The important feature of this transformation is that many relationships and changes that exist on the original function f(t) are also established in its transformed F(s) with a simple logical relationship.

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